Name of College - S. S. College,
J- Road

Deptt - Mathematics

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DATE: 26-08-2020

Topic - Integration of Rational Algebraic Function

Class - B.Sc. I (Hons)

Time - 10.15 to 11.00 A.M
11.00 to 11.45 A.M

Date - 26-08-2020

By - Dr. Shree Nath Sharma.

Problem. $\int \frac{dx}{x^6-1}$

$$= \int \frac{1}{(x^3-1)(x^3+1)} dx$$

$$= \frac{1}{2} \int \left[\frac{1}{x^3-1} - \frac{1}{x^3+1}\right] dx$$

$$= \frac{1}{2} \int \frac{dx}{x^3-1} - \frac{1}{2} \int \frac{dx}{x^3+1}$$

$$= \frac{1}{2} I_1 - \frac{1}{2} I_2$$

Now $I_1 = \int \frac{dx}{x^3-1}$

$$= \int \frac{dx}{(x-1)(x^2+x+1)}$$

Let $\frac{1}{x^3-1} = \frac{1}{(x-1)(x^2+x+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1}$

$$\Rightarrow 1 = A(x^2+x+1) + Bx(x-1) + C(x-1)$$

$$= Ax^2 + Ax + A + Bx^2 - Bx + Cx - C$$

$$= (A+B)x^2 + (A-B+C)x + A - C$$

Equating the co-efficient of x^2, x and constant

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We have $A+B=0$ —— (I)

$A-B+C=0$ —— (II)

$A-C=1$ —— (III)

(II) $\Rightarrow B=A+C$

From (I) $2A+C=0$

$A-C=1$

$3A=1$ Also $B=-A$

$A=1/3$ $=-1/3$

from (III) $A-1=C$

$1/3-1=C$

$-2/3=C$

$$\boxed{\frac{1}{6}-\frac{2}{3}=\frac{1-4}{6}=-\frac{1}{2}}$$

$$\therefore \frac{1}{x^3-1}=\frac{1}{(x-1)(x^2+x+1)}=\frac{1}{3}\frac{1}{(x-1)}+\frac{-\frac{1}{3}x-\frac{2}{3}}{x^2+x+1}$$

$$\therefore I_{x}=\int\frac{1}{x^3-1}dx=\frac{1}{3}\int\frac{dx}{x-1}-\frac{1}{3}\int\frac{x\,dx}{x^2+x+1}-\frac{2}{3}\int\frac{dx}{x^2+x+1}$$

$$=\frac{1}{3}\log(x-1)-\frac{1}{3\times2}\int\frac{2x+1-1}{x^2+x+1}dx-\frac{2}{3}\int\frac{dx}{x^2+x+1}$$

$$=\frac{1}{3}\log(x-1)-\frac{1}{6}\int\frac{2x+1}{x^2+x+1}dx-\frac{1}{2}\int\frac{dx}{x^2+x+1}$$

$$=\frac{1}{3}\log(x-1)-\frac{1}{6}\log(x^2+x+1)-\frac{1}{2}\int\frac{dx}{(x+\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2}$$

put $x+\frac{1}{2}=z$ $\frac{\sqrt{3}}{2}=k$

$dx=dz$

$$=\frac{1}{3}\log(x-1)-\frac{1}{6}\log(x^2+x+1)-\frac{1}{2}\int\frac{dz}{z^2+k^2}$$

$$=\frac{1}{3}\log(x-1)-\frac{1}{6}\log(x^2+x+1)-\frac{1}{2}\cdot\frac{1}{k}\tan^{-1}\frac{z}{k}$$

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Put $k = \sqrt{3}/2$

$z = x + 1/2$

$$= \frac{1}{3}\log(x-1) - \frac{1}{6}\log(x^2+x+1) - \frac{1}{2\times\frac{\sqrt{3}}{2}}\tan^{-1}\frac{x+1/2}{\sqrt{3}/2}$$

$$= \frac{1}{3}\log(x-1) - \frac{1}{6}\log(x^2+x+1) - \frac{1}{\sqrt{3}}\tan^{-1}\frac{2x+1}{\sqrt{3}} + k$$

$$I_2 = \int \frac{1}{x^3+1}\,dx$$

Let

$$\frac{1}{x^3+1} = \frac{1}{(x+1)(x^2-x+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2-x+1}$$

$$\therefore 1 = A(x^2-x+1) + Bx^2 + Bx + Cx + C$$

$$= (A+B)x^2 + (B-A+C)x + A + C$$

Equating co-efficient of x^2, x and constant we have

$A + B = 0$ ——— (i)

$B + C - A = 0$ ——— (ii)

$A + C = 1$ ——— (iii)

From (i) $\cancel{-A} = B$

using in (ii)

$2B + C = 0$

$B + C = 1$

On substraction

$3B = -1$

$B = -1/3 \quad \therefore A = 1/3$

$C = 1 - B$

$= 1 + 1/3 = 4/3$

(i) $\Rightarrow B = -A$

put in (ii)

$\cancel{C} - 2A = 0$

$\cancel{C} + A = 1$

$-3A = -1$

$A = 1/3$

$B = -1/3$

From (iii)

$A + C = 1$

$C = 1 - A$

$= 1 - 1/3$

$= 2/3$

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Thus we have $A = \frac{1}{3}$, $B = -\frac{1}{3}$, $C = \frac{2}{3}$

$$\therefore \frac{1}{x^3+1} = \frac{1}{(x+1)(x^2-x+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2-x+1}$$

$$= \frac{1}{3(x+1)} + \frac{-\frac{1}{3}x + \frac{2}{3}}{x^2-x+1}$$

$$= \frac{1}{3(x+1)} - \frac{1}{3\times 2}\frac{2x}{x^2-x+1} + \frac{2}{3(x^2-x+1)}$$

$$= \frac{1}{3(x+1)} - \frac{1}{6}\frac{2x-1+1}{x^2-x+1} + \frac{2}{3(x^2-x+1)}$$

$$= \frac{1}{3(x+1)} - \frac{1}{6}\frac{2x-1}{x^2-x+1} + \left(\frac{2}{3} - \frac{1}{6}\right)\frac{1}{x^2-x+1}$$

$$= \frac{1}{3(x+1)} - \frac{1}{6}\frac{2x-1}{x^2-x+1} + \frac{1}{2}\frac{1}{x^2-x+1}$$

Thus $\int \frac{1}{x^3+1}\,dx = \frac{1}{3}\int \frac{dx}{x+1} - \frac{1}{6}\int \frac{2x-1}{x^2-x+1}\,dx + \frac{1}{2}\int \frac{dx}{x^2-x+1}$

Since we know if N^r is Differential co-efficient of D^r then integral of such function is equal to log of D^r

$$= \frac{1}{3}\log(x+1) - \frac{1}{6}\log(x^2-x+1) + \frac{1}{2}\int \frac{dx}{(x-\frac{1}{2})^2 +}$$

$$= \frac{1}{3}\log(x+1) - \frac{1}{6}\log(x^2-x+1) + \frac{1}{\cancel{2}}\cdot\frac{1}{\frac{\sqrt{3}}{\cancel{2}}}\tan^{-1}\frac{x}{\sqrt{}}$$

$$= \frac{1}{3}\log(x+1) - \frac{1}{6}\log(x^2-x+1) + \frac{1}{\sqrt{3}}\tan^{-1}\frac{2x-1}{\sqrt{3}}$$

$$\therefore \int \frac{1}{x^6-1}\,dx = \frac{1}{2}\left[I_1 - I_2\right]$$

$$\Rightarrow \int \frac{1}{x^6-1}\,dx = \frac{1}{2}\left[\left\{\frac{1}{3}\log(x-1) - \frac{1}{6}\log(x^2+x+1) - \frac{1}{\sqrt{3}}\tan^{-1}\frac{2x+1}{\sqrt{3}}\right\} - \left\{\frac{1}{3}\log(x+1) - \frac{1}{6}\log(x^2-x+1) + \frac{1}{\sqrt{3}}\tan^{-1}\frac{2x+1}{\sqrt{3}}\right\}\right] + C$$

Form :→

$$\int \frac{x^2+1}{x^4+kx^2+1}\,dx \quad \text{and} \quad \int \frac{x^2-1}{x^4+kx^2+1}\,dx$$

Method Divide ~~by~~ N^r and D^r by x^2 then put $x - \frac{1}{x} = z$ in (I) and $x + \frac{1}{x} = z$ in (II)

Problem

* $$\int \frac{x^2+1}{x^4+1}\,dx$$

$$= \int \frac{x^2\left[1+\frac{1}{x^2}\right]}{x^2\left[x^2+\frac{1}{x^2}\right]}\,dx$$

$$= \int \frac{\left(1+\frac{1}{x^2}\right)dx}{x^2+\frac{1}{x^2}}$$

put $x - 1/x = z$

$\left[1+\frac{1}{x^2}\right]dx = dz$

$$\left(x-\frac{1}{x}\right)^2 = z^2$$

$$\Rightarrow x^2 + \frac{1}{x^2} - 2 = z^2$$

$$\Rightarrow x^2 + \frac{1}{x^2} = z^2 + 2$$

$$= \int \frac{dz}{z^2+2} = \int \frac{dz}{z^2+(\sqrt{2})^2}$$

$$= \frac{1}{\sqrt{2}}\tan^{-1}\frac{z}{\sqrt{2}}$$

$$\frac{1}{\sqrt{2}}\tan^{-1}\frac{x - 1/x}{\sqrt{2}} = \frac{1}{\sqrt{2}}\tan^{-1}\frac{x^2-1}{\sqrt{2}x} + C$$

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Problem 2. $\int \frac{x^2-1}{x^4+1}\,dx$

$$= \int \frac{x^2(1-1/x^2)}{x^2\left[x^2+\frac{1}{x^2}\right]}\,dx$$

$$= \int \frac{1-1/x^2}{\left[x+\frac{1}{x}\right]^2-2}\,dx$$

put $x+\frac{1}{x} = z$

$\left[1-\frac{1}{x^2}\right]dx = dz$

$$= \int \frac{dz}{z^2-(\sqrt{2})^2}$$

$$= \frac{1}{2\sqrt{2}} \log \frac{z-\sqrt{2}}{z+\sqrt{2}} + C$$

$$= \frac{1}{2\sqrt{2}} \log \frac{x+\frac{1}{x}-\sqrt{2}}{x+\frac{1}{x}+\sqrt{2}} + C$$

$$= \frac{1}{2\sqrt{2}} \log \frac{x^2-\sqrt{2}x+1}{x^2+\sqrt{2}x+1} + C$$

Problem $\rightarrow$ $\int \frac{dx}{x^4+1}$

$$= \frac{1}{2}\int \frac{(x^2+1)-(x^2-1)}{x^4+1}\,dx$$

$$= \frac{1}{2}\int \frac{x^2+1}{x^4+1}\,dx - \frac{1}{2}\int \frac{x^2-1}{x^4+1}\,dx$$

$$= \frac{1}{2}I_1 - \frac{1}{2}I_2$$

Problem $\int \frac{x^2-1}{x^4+x^2+1}\,dx$

$$= \int \frac{x^2\left(1-\frac{1}{x^2}\right)}{x^2\left(x^2+\frac{1}{x^2}+1\right)}\,dx$$

$$= \int \frac{1-1/x^2}{(x+1/x)^2-1}\,dx$$

put $x+1/x = z \Rightarrow (1-1/x^2)\,dx = dz$

$$= \int \frac{dz}{z^2-1}$$

$$= \frac{1}{2}\log\frac{z-1}{z+1} + c$$

$$= \frac{1}{2}\log\frac{x+1/x-1}{x+1/x+1} + c$$

$$= \frac{1}{2}\log\frac{x^2-x+1}{x^2+x+1} + c.$$

Problem. $\int \frac{x^2+1}{x^4+x^2+1}\,dx$

$$= \int \frac{x^2\left(1+1/x^2\right)}{x^2\left(x^2+1+1/x^2\right)}\,dx$$

$$= \int \frac{1+1/x^2}{(1-1/x)^2+3}\,dx$$

Put $x - \frac{1}{x} = z$

$$\left(1 + \frac{1}{x^2}\right) dx = dz$$

$$\therefore I = \int \frac{dz}{z^2 + 3}$$

$$= \int \frac{dz}{z^2 + (\sqrt{3})^2}$$

$$= \frac{1}{\sqrt{3}} \tan^{-1} \frac{z}{\sqrt{3}} + C$$

$$= \frac{1}{\sqrt{2}} \tan^{-1} \frac{x - \frac{1}{x}}{\sqrt{3}} + C$$

$$= \frac{1}{\sqrt{2}} \tan^{-1} \frac{x^2 - 1}{\sqrt{3}\, x} + C$$

Problem $I = \int \frac{x^2}{x^4 + x^2 + 1}$

$$= \frac{1}{2} \int \frac{(x^2+1) + (x^2-1)}{x^4 + x^2 + 1} dx$$

$$= \frac{1}{2} \int \frac{x^2+1}{x^4+x^2+1} dx + \frac{1}{2} \int \frac{x^2-1}{x^4+x^2+1} dx$$

Problem $\int \frac{dx}{x^4 + a^4}$

$$= \frac{1}{2a^2} \int \frac{(x^2+a^2) - (x^2-a^2)}{x^4+a^4} dx$$

$$= \frac{1}{2a^2}\int \frac{x^2+a^2}{x^4+a^4}\,dx - \frac{1}{2a^2}\int \frac{x^2-a^2}{x^4+a^4}\,dx$$

$$= \frac{1}{2a^2} I_1 - \frac{1}{2a^2} I_2$$

Where $I_1 = \int \frac{x^2+a^2}{x^4+a^4}\,dx$ $\qquad I_2 = \int \frac{x^2-a^2}{x^4+a^4}$

$$= \int \frac{x^2\left(1+a^2/x^2\right)}{x^2\left[x^2+\frac{a^4}{x^2}\right]}\,dx$$

$$= \int \frac{1+a^2/x^2}{x^2+\frac{a^4}{x^2}}\,dx$$

$$= \int \frac{1+a^2/x^2}{(x-a^2/x)^2+2a^2}\,dx$$

put $x - a^2/x = z$

$1+\frac{a^2}{x^2}\,dx = dz$

$$= \int \frac{dz}{z^2+(\sqrt{2}a)^2}$$

$$= \frac{1}{\sqrt{2}a}\tan^{-1}\frac{z}{\sqrt{2}a}$$

$$= \frac{1}{\sqrt{2}a}\tan^{-1}\frac{x-a^2/x}{\sqrt{2}a}$$

$$= \frac{1}{\sqrt{2}a}\tan^{-1}\frac{x^2-a^2}{\sqrt{2}ax}$$

$$= \int \frac{x^2\left(1-a^2/x^2\right)}{x^2\left[x^2+\frac{a^4}{x^2}\right]}\,dx$$

$$= \int \frac{1-a^2/x^2}{\left(x+\frac{a^2}{x}\right)^2-2a^2}\,dx$$

put $x + a^2/x = z$

$\left[1-\frac{a^2}{x^2}\right]dx = dz$

$$= \int \frac{dz}{z^2-(\sqrt{2}a)^2}$$

$$= \frac{1}{2a\sqrt{2}}\log\frac{z-\sqrt{2}a}{z+\sqrt{2}a}$$

$$= \frac{1}{2\sqrt{2}a}\log\frac{x+\frac{a^2}{x}-\sqrt{2}a}{x+a^2/x+\sqrt{2}a}$$

$$= \frac{1}{2\sqrt{2}a}\log\frac{x^2+a^2-\sqrt{2}ax}{x^2+a^2+\sqrt{2}ax}$$

thus $I = \frac{1}{2a^2}\left[I_1 - I_2\right]$